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If z is less than this, y is less than $\frac{a}{6}\sqrt{3(1+\sin\theta \operatorname{cosec}\phi)}$.

If $z > \frac{a}{3}(1+\sin\phi \operatorname{cosec}\theta)$ and $y < \frac{a}{6}\sqrt{3(1+\sin\theta \operatorname{cosec}\phi)}$ the limits of z are $\frac{a}{3}(1+\sin\phi \operatorname{cosec}\theta)$ and a .

$$\begin{aligned} \Delta = \Delta \left[\int_0^{\frac{1}{2}\pi} \frac{1}{3^2} \sin\theta \operatorname{cosec}^3\phi - \frac{1}{4^8} \sin^2\theta \operatorname{cosec}^4\phi + \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{1}{4^8} \sin^2\theta \operatorname{cosec}^4\phi - \frac{1}{2^7} \frac{1}{3^7} \right. \\ \times (\sin^2\theta \operatorname{cosec}^4\phi + \sin^2\phi \operatorname{cosec}^2\theta) + \frac{1}{2^5} \frac{1}{3^6} (\sin\theta \operatorname{cosec}^3\phi + \sin\phi \operatorname{cosec}^3\theta + \frac{41}{2^4} \frac{1}{3^7} \\ \left. \times \operatorname{cosec}\theta \operatorname{cosec}\phi \right] d\theta \div \frac{1}{3^2} \int_0^{\frac{1}{2}\pi} \sin\theta \operatorname{cosec}^3\phi d\theta = \frac{\Delta}{3^6} (470 + \frac{4}{3} \log 2) = .6997 \Delta. \end{aligned}$$

This is problem 76, p. 513, Williamson's *Integral Calculus*.

170. Proposed by LON C. WALKER, A. M., Santa Barbara, California.

Find the area of a triangle formed by drawing a line at random through each of three points taken at random within a given triangle.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

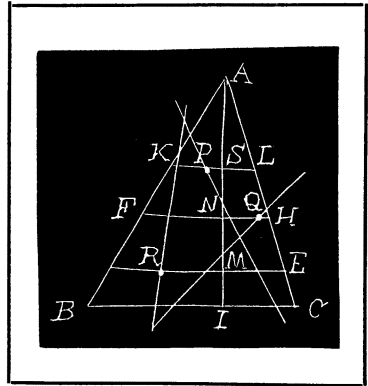
Let ABC be the given triangle; P, Q, R the random points; $AI=h$, $BI=d$, $CI=e$, $d+e=a$, $AM=u$, $MR=v$, $AS=w$, $SP=z$, $AN=m$, $NQ=n$. $y-v=r(x-u)$, the line through R ... (1), $y-z=s(x-w)$, the line through P ... (2), $y-n=t(x-m)$, the line through Q ... (3), where $r=\tan\theta$, $s=\tan\phi$, $t=\tan\psi$.

The intersection of (1) and (2) is given by

$$x_1 = \frac{ru - ws + z - v}{r - s}, \quad y_1 = \frac{rsu - rsw + rz - sv}{r - s}.$$

The intersection of (1) and (3) is given by

$$x_2 = \frac{ru - mt + n - v}{r - t}, \quad y_2 = \frac{rtu - mrt + rn - tv}{r - t}.$$



The intersection of (2) and (3) is given by

$$x_3 = \frac{sw - mt + n - z}{s - t}, \quad y_3 = \frac{stw - mst + sn - tz}{s - t}.$$

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2}(x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1) \\ &= \frac{1}{2}[(ur-v)(s-t) + (ws-z)(t-r) + (mt-n)(r-s)]^2 / (r-s)(r-t)(s-t) \\ &= A/B. \end{aligned}$$

The limits of u are 0 and h ; of w , 0 and u ; of m , w and u ; of v , $-du/h=v_1$ and $eu/h=v_2$; of z , $-dw/h=w_1$ and $ew/h=w_2$; of n , $-dm/h=m_1$ and $em/h=m_2$. The number of ways the three points can be taken on the surface of the triangle is $\frac{1}{8}(\frac{1}{2}ah)^3 = \frac{1}{48}a^3h^3$.

Hence if Δ = the required average area we get

$$\begin{aligned}
 \Delta &= \frac{48}{a^3 h^3} \int_0^h \int_0^u \int_w^u \int_{v_i}^{v_2} \int_{w_1}^{w_2} \int_{n_1}^{n_2} \frac{A}{B} du dw dm dv dz dn \\
 &= \frac{8}{ah^3 B} \int_0^h \int_0^u \int_w^u \{ (s-t)^2 [3ah^2 r^2 + 3hr(d^2 - e^2) + d^3 + e^3] u^3 wm \\
 &\quad + (t-r)^2 [3ah^2 s^2 + 3hs(d^2 - e^2) + d^3 + e^3] uw^3 m \\
 &\quad + (r-s)^2 [3ah^2 t^2 + 3ht(d^2 - e^2) + d^3 + e^3] uwm^3 \} du dw dm \\
 &+ \frac{12}{a^2 h^3 B} \int_0^h \int_0^u \int_w^u [(s-t)(t-r)(2ahr + d^2 - e^2)(2ahs + d^2 - e^2) u^2 w^2 m \\
 &\quad + (s-t)(r-s)(2ahr + d^2 - e^2)(2akt + d^2 - e^2) u^2 wm^2 \\
 &\quad + (t-r)(r-s)(2akt + d^2 - e^2)(2ahs + d^2 - e^2) uw^2 m^2] du dw dm \\
 &= \frac{1}{24a} \left[\frac{9ah^2 r^2 (s-t)}{(r-s)(r-t)} + \frac{9hr(s-t)(d^2 - e^2)}{(r-s)(r-t)} + \frac{3(d^3 + e^3)(s-t)}{(r-s)(r-t)} \right. \\
 &\quad + \frac{3ah^2 s^2 (r-t)}{(r-s)(s-t)} + \frac{3hs(r-t)(d^2 - e^2)}{(r-s)(s-t)} + \frac{(d^3 + e^3)(r-t)}{(r-s)(s-t)} \\
 &\quad + \frac{6ah^2 t^2 (r-s)}{(s-t)(r-t)} + \frac{6ht(r-s)(d^2 - e^2)}{(s-t)(r-t)} + \left. \frac{2(d^3 + e^3)(r-s)}{(s-t)(r-t)} \right] \\
 &+ \frac{1}{60a^2} \left[\frac{24a^2 h^2 rs}{s-r} + \frac{36a^2 h^2 rt}{r-t} + \frac{20a^2 h^2 st}{t-s} + \frac{12ah(d^2 - e^2)(r+s)}{s-r} \right. \\
 &\quad + \frac{18ah(d^2 - e^2)(r+t)}{r-t} + \frac{10ah(d^2 - e^2)(s+t)}{t-s} + \frac{6(d^2 - e^2)}{s-r} + \frac{9(d^2 - e^2)^2}{r-t} \\
 &\quad + \left. \frac{5(d^2 - e^2)^2}{t-s} \right] = \frac{1}{24a} C + \frac{1}{60a^2} D.
 \end{aligned}$$

The limits of θ , ϕ , and ψ are for each 0 and $\frac{1}{2}\pi$ and doubled.

$$\text{Now } \frac{r^2(s-t)}{(r-s)(r-t)} = \frac{r^2}{r-s} - \frac{r^2}{r-t}$$

$$\begin{aligned}
& \therefore \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{r^2 (s-t)}{(r-s)(r-t)} d\theta d\phi d\psi = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{s^2 (r-t)}{(r-s)(s-t)} d\theta d\phi d\psi \\
& = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{t^2 (r-s)}{(s-t)(r-t)} d\theta d\phi d\psi = 0. \\
& \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{r(s-t)}{(r-s)(r-t)} d\theta d\phi d\psi = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{s(r-t)}{(r-s)(s-t)} d\theta d\phi d\psi \\
& = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{t(r-s)}{(s-t)(r-t)} d\theta d\phi d\psi = 0. \\
& \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{(s-t)}{(r-s)(r-t)} d\theta d\phi d\psi = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{(r-t)}{(r-s)(s-t)} d\theta d\phi d\psi \\
& = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{(r-s)}{(s-t)(r-t)} d\theta d\phi d\psi = 0. \quad \therefore C=0. \\
& \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{rs}{r-s} d\theta d\phi d\psi = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{rt}{r-t} d\theta d\phi d\psi \\
& = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{st}{t-s} d\theta d\phi d\psi = \frac{1}{8}\pi^2. \\
& \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{r+s}{r-s} d\theta d\phi d\psi = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{r+t}{r-t} d\theta d\phi d\psi \\
& = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{s+t}{t-s} d\theta d\phi d\psi = 0. \\
& \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{d\theta d\phi d\psi}{s-r} = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{d\theta d\phi d\psi}{r-t} = \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{d\theta d\phi d\psi}{t-s} = -\frac{1}{8}\pi^2. \\
& \therefore D = \frac{10\pi^2 a^2 h^2 - \frac{5}{2}\pi^2 (d^2 - e^2)^2}{\frac{1}{8}\pi^3}. \\
& \therefore \Delta = \frac{4a^2 h^2 - (d^2 - e^2)^2}{3\pi a^2} = \frac{[a^2 (2c^2 + 2b^2 - a^2) - 2(c^2 - b^2)^2]}{3\pi a^2}. \\
& \text{If } a=b=c, \Delta = a^2/\pi.
\end{aligned}$$